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Conservative Scheme for a Model of Nonlinear Dispersive Waves and Its Solitary Waves Induced by Boundary Motion

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A conservative difference scheme is given for a model of nonlinear dispersive waves. Convergence and stability of the scheme are proved. By means of this scheme, we explore numerically the relationship between the boundary data and the amplitudes and number of solitary waves it produces.

1. INTRODUCTION

In recent years, a vast amount of work and computation has been devoted to the initial value problem for the KdV equation. Under the assumption of small amplitude and large wavelength, the KdV equation was derived for water waves and it is similarly justifiable as a model for long wave in many other physical systems. However, in view of the order of magnitude of the terms in KdV equation, another equation for nonlinear dispersive waves can be derived,

$$U_t + U_x + \beta U U_x - \gamma^{-2} \cdot U_{xxt} = 0, \tag{1.1}$$

where $\beta \geqslant 0$ and $\gamma > 0$ are constants. Eq. (1.1) and the KdV equation are advocated as models for the same physical phenomena and are valid to the same accuracy.

Eq. (1.1) has been studied by several workers. Mathematical theory for the equation was considered in [1, 2, 5]. Bona *et al.* [2] have compared the equation with the results of some experiments. Several numerical methods for solving Eq. (1.1) have been developed [4–8, 10]. In this paper, we present a conservative difference scheme for Eq. (1.1), which keeps two conservation laws that the differential equation (1.1) possesses. The conservative property of the difference scheme is signifi-

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cant, especially, when solitary solutions are studied numerically. Convergence and stability of the scheme are proved.

Solitary waves induced by boundary motion were studied for the KdV equation in Chu *et al.* [3]. Their results are interesting, and we shall also study the solitary waves induced by boundary motion for Eq. (1.1), by means of our conservative difference scheme.

2. Numerical Method

We consider the initial-boundary value problem for the model of nonlinear dispersive waves

$$U_t + U_x + \beta U U_x - \gamma^{-2} U_{xxt} = 0, \qquad 0 < x < x_L,\ t > 0, \tag{2.1}$$

$$U|_{x=0} = 0, \qquad U|_{x=x_L} = 0, \qquad t > 0, \tag{2.2}$$

$$U|_{t=0} = U_0(x), \qquad 0 < x < x_L. \tag{2.3}$$

As usual, the following notations are used

$$U_j^n \sim U(x_j, t^n), \qquad x = jh,\ t^n = n \cdot \tau,\ 0 \leqslant j \leqslant J,\ n = 0, 1, 2, ...,$$

$$(U_j^n)_x = \frac{1}{h}(U_{j+1}^n - U_j^n), \qquad (U_j^n)_{\bar{x}} = \frac{1}{h}(U_j^n - U_{j-1}^n),$$

$$(U_j^{n+1})_t = \frac{1}{\tau}(U_j^{n+1} - U_j^n), \qquad (U_j^n)_{\hat{x}} = \frac{1}{2h}(U_{j+1}^n - U_{j-1}^n),$$

$$U_j^{n+1/2} = \frac{1}{2}(U_j^{n+1} + U_j^n), \qquad \|U\|_{L_2}^2 = \int_0^{x_L} |U(x, t)|^2\, dx,$$

$$\|U^n\|^2 = h \sum_{j=1}^{J-1} |U_j^n|^2, \qquad \|U_x^n\|^2 = h \sum_{j=0}^{J-1} |(U_j^n)_x|^2,$$

$$\|U^n\|_\infty = \sup_{1 \leqslant j \leqslant J-1} |U_j^n|,$$

where $h = X_L/J$ and τ are step sizes of space and time, respectively.

Multiplying (2.1) by U and integrating it from 0 to X_L, it is easy to get a conservation law for the problem (2.1)–(2.3)

$$E(t) = \|U\|_{L_2}^2 + \gamma^{-2} \cdot \|U_x\|_{L_2}^2 = \|U_0(x)\|_{L_2}^2 + \gamma^{-2} \cdot \left\|\frac{dU_0(x)}{dx}\right\|_{L_2}^2 = \text{Const.} \tag{2.4}$$

Using a customary designation, we shall refer to the functional $E(t)$ as the energy integral, although it is not necessarily identifiable with energy in the original physical problem.

Our proposed difference scheme for problem (2.1)–(2.3) is

$$(U_j^{n+1})_{\bar{t}}+(U_j^{n+1/2})_{\hat{x}}-\gamma^{-2}(U_j^{n+1})_{x\bar{x}\bar{t}}+\frac{\beta}{3}\{U_j^{n+1/2}\cdot(U_j^{n+1/2})_{\hat{x}}+[(u_j^{n+1/2})^2]_{\hat{x}}\}$$
$$=0, \qquad 1\leqslant j\leqslant J-1,\ n=0,1,2,..., \tag{2.5}$$

$$U_0^n=0, \qquad U_J^n=0 \tag{2.6}$$

$$U_j^0=U_0(x_j), \qquad 1\leqslant j\leqslant J-1. \tag{2.7}$$

Multiplying (2.5) by $2\cdot U_j^{n+1/2}$ and summing over j, we have

$$\frac{1}{\tau}\sum_{j=1}^{J-1}[(U_j^{n+1})^2-(U_j^n)^2]+2\sum_{j=1}^{J-1}(U_j^{n+1/2})_{\hat{x}}\cdot U_j^{n+1/2}$$
$$-2\gamma^{-2}\sum_{j=1}^{J-1}(U_j^{n+1})_{x\bar{x}\bar{t}}\cdot U_j^{n+1/2}$$
$$+\frac{2\beta}{3}\left\{\sum_{j=1}^{J-1}(U_j^{n+1/2})^2\cdot(U_j^{n+1/2})_{\hat{x}}+\sum_{j=1}^{J-1}[(U_j^{n+1/2})^2]_{\hat{x}}\cdot U_j^{n+1/2}\right\}$$
$$=0, \qquad 1\leqslant j\leqslant J-1,\ n=0,1,2,..., \tag{2.8}$$

In view of difference properties and the boundary conditions (2.6), we obtain

$$\sum_{j=1}^{J-1}(U_j^{n+1/2})_{\hat{x}}\cdot U_j^{n+1/2}=\frac{1}{2h}\sum_{j=1}^{J-1}[U_{j+1}^{n+1/2}\cdot U_j^{n+1/2}-U_{j-1}^{n+1/2}\cdot U_j^{n+1/2}]$$
$$=\frac{1}{2h}\left[\sum_{j=0}^{J-1}U_{j+1}^{n+1/2}\cdot U_j^{n+1/2}-\sum_{j=1}^{J}U_{j-1}^{n+1/2}\cdot U_j^{n+1/2}\right]=0,$$

$$\sum_{j=1}^{J-1}(U_j^{n+1})_{x\bar{x}\bar{t}}\cdot U_j^{n+1/2}=\frac{1}{h}\left[\sum_{j=1}^{J-1}(U_j^{n+1})_{x\bar{t}}\cdot U_j^{n+1/2}-(U_{j-1}^{n+1})_{x\bar{t}}\cdot U_j^{n+1/2}\right]$$
$$=\frac{1}{h}\left[\sum_{j=1}^{J-1}(U_j^{n+1})_{x\bar{t}}\,U_j^{n+1/2}-\sum_{j=0}^{J-1}(U_j^{n+1})_{x\bar{t}}\cdot U_{j+1}^{n+1/2}\right]$$
$$=\frac{1}{h}\left[\sum_{j=0}^{J-1}(U_j^{n+1})_{x\bar{t}}\cdot U_j^{n+1/2}-\sum_{j=0}^{J-1}(U_j^{n+1})_{x\bar{t}}\cdot U_{j+1}^{n+1/2}\right]$$
$$=-\sum_{j=0}^{J-1}(U_j^{n+1})_{x\bar{t}}\cdot(U_j^{n+1/2})_x$$
$$=-\frac{1}{2\tau}\sum_{j=0}^{J-1}\{[(U_j^{n+1})_x]^2-[(U_j^n)_x]^2\},$$

$$\sum_{j=1}^{J-1} (U_j^{n+1/2})^2 \cdot (U_j^{n+1/2})_{\hat{x}} = \frac{1}{2h} \sum_{j=1}^{J-1} [(U_j^{n+1/2})^2 \cdot U_{j+1}^{n+1/2} - (U_j^{n+1/2})^2 \cdot U_{j-1}^{n+1/2}]$$

$$= \frac{1}{2h} \left[\sum_{j=1}^{J-1} (U_{j-1}^{n+1/2})^2 \cdot U_j^{n+1/2} - \sum_{j=1}^{J-1} (U_{j+1}^{n+1/2})^2 \cdot U_j^{n+1/2} \right]$$

$$= -\sum_{j=1}^{J-1} [(U_j^{n+1/2})^2]_{\hat{x}} \cdot U_j^{n+1/2}.$$

Combining this result with (2.8) yields

$$\frac{1}{\tau} \sum_{j=1}^{J-1} [(U_j^{n+1})^2 - (U_j^n)^2] + \gamma^{-2} \cdot \frac{1}{\tau} \sum_{j=0}^{J-1} \{[(U_j^{n+1})_x]^2 - [(U_j^n)_x]^2\} = 0,$$

$$n = 0, 1, 2, \ldots$$

i.e.,

$$\|U^{n+1}\|^2 + \gamma^{-2} \|U_x^{n+1}\|^2 = \|U^0\|^2 + \gamma^{-2} \|U_x^0\|^2$$

$$= \text{Const.}, \qquad n = 0, 1, 2, \ldots \tag{2.9}$$

Comparing (2.9) with the conservation law of energy (2.4) that the differential problem (2.1)–(2.3) possesses, we can see the scheme (2.5)–(2.7) preserves a discrete analog of this conservation law. For this reason we call the scheme (2.5)–(2.7) conservative. Furthermore, we consider the initial value problem for Eq. (1.1)

$$U_t + U_x + \beta U U_x - \gamma^2 U_{xxt} = 0, \qquad -\infty < x < \infty,\ t > 0, \tag{2.10}$$

$$U|_{t=0} = U_0(x), \qquad -\infty < x < \infty. \tag{2.11}$$

Assume U, U_x, $U_t \to 0$ when $|x| \to \infty$. It is obvious that the conservation law of energy is valid for the problem (2.10), (2.11) and the corresponding difference problem (2.5), (2.7). Integrating (2.10) for x, we obtain another conservation law for the problem (2.10), (2.11),

$$\int_{-\infty}^{\infty} U(x, t)\, dx = \int_{-\infty}^{\infty} U_0(x)\, dx = \text{Const.} \tag{2.12}$$

Summing (2.5) over j we find

$$\sum_{j=-\infty}^{\infty} U_j^{n+1} = \sum_{j=-\infty}^{\infty} U_0(x_j) = \text{Const.}. \tag{2.13}$$

Therefore, for the initial value problem (2.10), (2.11), the conservative scheme preserves discrete versions of two basic conservation laws. The difference scheme

(2.5)–(2.7) is a system of nonlinear algebraic equations. We give an iterative method to solve it by the formulas

$$A_j^{n+1(s)} \cdot U_{j+1}^{n+1(s+1)} + B_j^{n+1(s)} \cdot U_j^{n+1(s+1)} + C_j^{n+1(s)} \cdot U_{j-1}^{n+1(s+1)} = F_j^{n+1(s)}, \qquad 1 \leqslant j \leqslant J-1,\ n = 0, 1, 2, \ldots, \tag{2.14}$$

$$U_0^n = 0, \qquad U_J^n = 0, \qquad n = 0, 1, 2, \ldots, \tag{2.15}$$

$$U_j^0 = U_0(x_j), \qquad U_j^{n+1(0)} = U_j^n, \qquad 1 \leqslant j \leqslant J-1,\ n = 0, 1, 2, \ldots, \tag{2.16}$$

where

$$A_j^{n+1(s)} = -\frac{1}{h^2} + \frac{\tau}{4h} + \frac{\tau}{24 \cdot h}\left(U_{j+1}^{n+1(s)} + U_{j+1}^n\right),$$

$$B_j^{n+1(s)} = 1 + \frac{2}{h^2} + \frac{\tau}{24 \cdot h}\left(U_{j+1}^{n+1(s)} + U_{j+1}^n\right) - \frac{\tau}{24 \cdot h}\left(U_{j-1}^{n+1(s)} + u_{j-1}^n\right),$$

$$C_j^{n+1(s)} = -\frac{1}{h^2} - \frac{\tau}{4h} - \frac{\tau}{24 \cdot h}\left(U_{j-1}^{n+1(s)} + U_{j-1}^n\right),$$

$$F_j^{n+1(s)} = \left[-\frac{1}{h^2} - \frac{\tau}{4h} - \frac{\tau}{24 \cdot h}\left(U_{j+1}^{n+1(s)} + U_{j+1}^n\right)\right] \cdot U_{j+1}^n + \left[1 + \frac{2}{h^2} - \frac{\tau}{24 \cdot h}\left(U_{j+1}^{n+1(s)} + U_{j+1}^n\right) + \frac{\tau}{24 \cdot h}\left(U_{j-1}^{n+1(s)} + U_{j-1}^n\right)\right] \times U_j^n + \left[-\frac{1}{h^2} + \frac{\tau}{4h} + \frac{\tau}{24 \cdot h}\left(U_{j-1}^{n+1(s)} + U_{j-1}^n\right)\right] \cdot U_{j-1}^n.$$

The time needed to compute $A_j^{n+1(s)}$, $B_j^{n+1(s)}$, $C_j^{n+1(s)}$, and $F_j^{n+1(s)}$ can be reduced by not unnecessarily computing quantities like

$$\frac{\tau}{24 \cdot h}\left(U_{j+1}^{n+1(s)} + U_{j+1}^n\right) \quad \text{and} \quad \frac{\tau}{24 \cdot h}\left(U_{j-1}^{n+1(s)} + U_{j-1}^n\right);$$

they are not recomputed unnecessarily. Equations (2.14)–(2.16) are a system of linear tridiagonal equations for $U_j^{n+1(s+1)}$, aftger $U_j^{n+1(s)}$ are obtained. Hence, $U_j^{n+1(s+1)}$ can be obtained as the formulas

$$\begin{aligned} &U_j^{n+1(s+1)} = \xi_j U_{j+1}^{n+1(s+1)} + \eta_j, \qquad j = J-1, J-2, \ldots, 1, \\ &U_J^{n+1(s+1)} = 0, \end{aligned} \tag{2.17}$$

where

$$\xi_j = \frac{-A_j^{n+1(s)}}{B_j^{n+1(s)} + C_j^{n+1(s)} \cdot \xi_{j-1}}, \qquad \eta_j = \frac{F_j^{n+1(s)} - C_j^{n+1(s)} \cdot \eta_{j-1}}{B_j^{n+1(s)} + C_j^{n+1(s)} \cdot \xi_{j-1}}, \qquad j = 1, 2, \ldots, J-1,$$

$$\xi_0 = 0, \qquad \eta_0 = 0.$$

Our experience has been that the iterative method (2.14)–(2.17) is quickly convergent for solving several systems of nonlinear equations, which are deduced from the differential equations.

3. Solitary Waves Induced by Boundary Motion

Until now, most work has been devoted to the behavior of solitary waves in unbounded domains. However, the mathematical models for many real physical phenomena are precisely initial boundary value problems of partial differential equations. For example, the production and propagation of water waves in a channel belongs to this case. Now we study these problems numerically and hope that interesting physical phenomena and mathematical properties can be uncovered by means of computational results.

In this section, the initial boundary value problem (2.1)–(2.3) is modified to

$$U_t + U_x + UU_x - U_{xxt} = 0, \qquad 0 < x < x_L,\ t > 0, \tag{3.1}$$

$$U|_{x=0} = f_1(t), \qquad U|_{x=x_L} = 0, \qquad t > 0, \tag{3.2}$$

$$U|_{t=0} = U_0(x), \qquad 0 < x < x_L. \tag{3.3}$$

and right boundary x_L is taken large enough to ignore its influence. Our intention is to explore the relationship between the amplitudes and numbers of the solitary waves produced and the boundary data given.

First, we take $U_0(x) \equiv 0$ and compute the solitary waves produced by a boundary pulse $f_1(t)$ of identical amplitude $A_0 = 2$ and duration $\Delta t = 20$ (See Fig. 1).

In computations, we take $h = 0.4$, $\tau = 0.1$ and the results are shown in Table I.

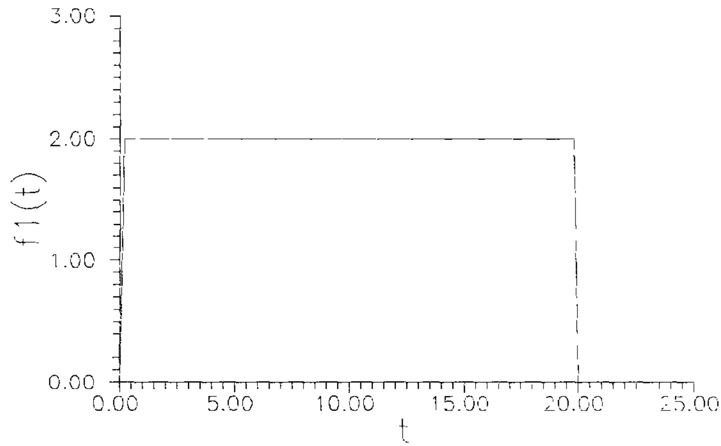

Fig. 1. Boundary pulse $f_1(t)$ given at $x = 0$.

TABLE I

Solitary Waves Induced by a Boundary Pulse

	Amplitude of solitary wave				
t	1	2	3	4	5
2.5	2.2337				
5.0	1.8614	2.7619			
10.0	2.5218	3.2387			
20.0	1.8560	2.3678	2.7327	3.4381	
40.0	0.9804	2.3173	3.0720	3.4893	3.7065
60.0	0.9803	2.3154	3.0681	3.5108	3.7478
80.0	0.9802	2.3146	3.0665	3.5128	3.7622
100.0	0.9801	2.3140	3.0660	3.5120	3.7620

Figure 2 shows the solitary waves at $t=100$. There are five solitary waves in this case. The amplitude of the last solitary wave is less than 2, since the boundary pulse was cut off during its formation at $t=20$.

Velocities of computational solitary waves and their relationship with the amplitudes are given in Table II.

The single solitary wave solution of Eq. (3.1) has the form

$$U=A\cdot \operatorname{Sech}^2(kx-\omega t+\delta),$$

where $A=3a^2/(1-a^2)$, $K=\frac{1}{2}a$, $\omega=a/2(1-a^2)$, and a and δ are arbitrary constants. It is clear that the velocity v of the solitary wave can be written as

$$v=\frac{\omega}{k}=\frac{1}{1-a^2}=1+\frac{A}{3}.$$

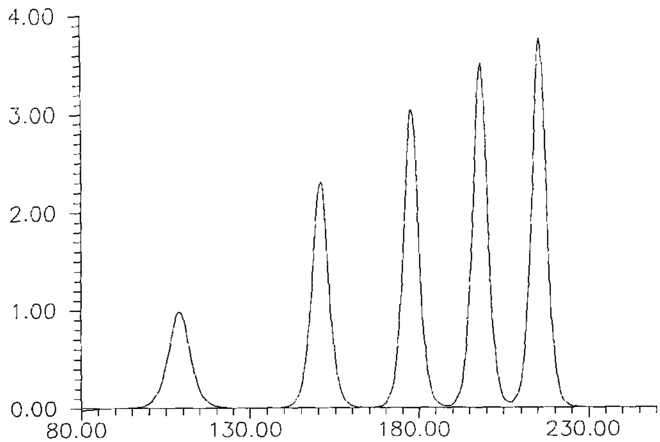

FIG. 2. Solitary waves at $t=100$.

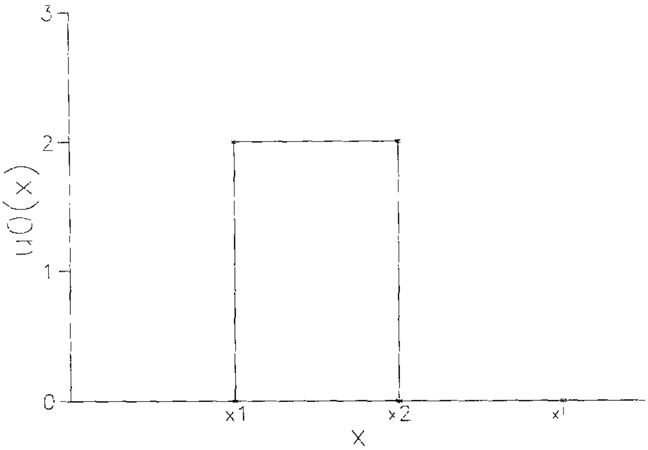

FIG. 3. Initial pulse $U_0(x)$ given at $t=0$.

Table II shows that the velocity of every solitary wave is equal to one plus a third of its amplitude. This velocity is precisely the velocity of a single solitary wave of sech-squared shape.

For the identical pulse amplitude $A_0=2$, solitary waves for various durations are given in Table III. When duration is fixed at $\Delta t=20$, solitary waves for several pulse amplitudes are given in Table IV.

As a comparison, we take $f_1(t)\equiv 0$ and compute the solitary waves produced by initial value $U_0(x)$. The pulse $U_0(x)$ is located in interval $[x_1, x_2]$ and has identical amplitude $A_0=2$ (see Fig. 3). Solitary waves for various initial pulses are given in Table V. Figure 4 shows the solitary waves for $x_1=20.0$, $x_2=31.6$.

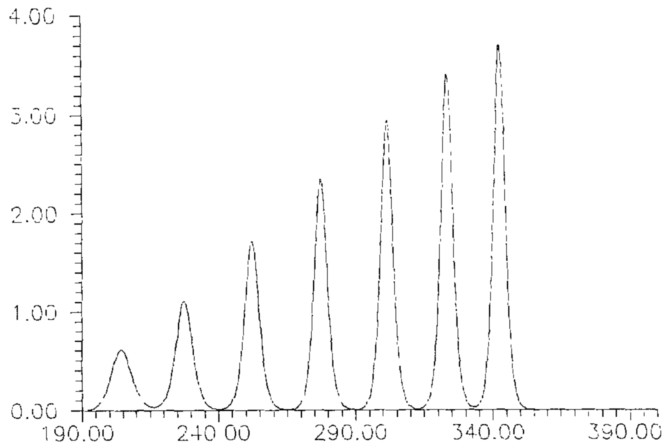

FIG. 4. Solitary waves produced by initial value $U_0(x)$ and $x_1=20.0$, $x_2=31.6$.

TABLE II

Velocities of Solitary Waves for $t \geqslant 40$

Solitary wave	Amplitude A	$1+A/3$	Velocity
1	0.98	1.3266	1.33
2	2.31	1.7700	1.77
3	3.07	2.0233	2.02
4	3.51	2.1700	2.17
5	3.76	2.2533	2.25

TABLE III

Solitary Waves for Various Durations of the Boundary Value Pulse with $A_0 = 2.0$

Duration Δt	Time t	Amplitude of solitary wave 1	2	3	4	5	6	7	8	9	10	11	12
50.0	200.0	0.6753	1.5216	2.0845	2.5240	2.8351	3.1618	3.3532	3.4816	3.5965	3.6987	3.7844	3.8643
35.0	150.0	0.1652	1.3704	2.1522	2.6881	3.0738	3.3485	3.5843	3.6902	3.8110			
20.0	100.0	0.9801	2.3140	3.0660	3.5120	3.7620							
10.0	100.0	0.4773	2.5906	3.5855									
4.0	100.0	2.5164											
3.0	100.0	1.8588											
2.0	30.0	1.0842[a]											

[a] There is not stable solitary wave.

TABLE IV

Solitary Waves for Several Amplitudes of the Boundary Value Pulse with $\Delta t = 20.0$

Pulse amplitude A_0	Time t	Amplitude of solitary wave 1	2	3	4	5	6	7	8	9	10
5.0	100.0	0.7249	4.4651	5.9871	6.9481	7.7501	8.3676	8.8621	9.1220	9.2356	9.3393
1.0	100.0	0.8056	1.4709	1.8314							
0.5	100.0	0.4962	0.8457								

TABLE V

Solitary Waves for Various Initial Pulses

Initial pulse x_1	Initial pulse x_2	Time t	Amplitude of solitary wave 1	2	3	4	5	6	7	8	9
20.0	36.0	100.0	0.6253	1.0512	1.5236	2.0074	2.4742	2.9143	3.2368	3.5669	3.7551
20.0	31.6	100.0	0.6151	1.0810	1.7078	2.3437	2.9221	3.4081	3.7291		
20.0	30.0	100.0	0.6054	1.2781	2.0012	2.7025	3.2974	3.6963			

These tables and figures demonstrates a steady non-oscillating, long pulse at the boundary can produce a train of stable solitary waves whose amplitudes progressively decrease. The amplitudes of the induced waves are governed mainly by the amplitude of the boundary pulse, while the number of the induced waves depends on the duration and amplitude of the boundary pulse. Furthermore, the initial pulse can also produce a train of stable solitary waves, which is similar to that produced by the boundary pulse.

4. Discussion for Numerical Method

(i) *Convergence and Stability of the Difference Scheme*

Now, we consider convergence and stability of the conservative difference scheme (2.5)–(2.7).

Lemma 1 (discrete Sobolev inequality [9]). *There are constants C_1 and C_2 such that*

$$\|U^n\|_\infty \leqslant C_1 \|U^n\| + C_2 \|U_x^n\|.$$

Lemma 2. *Assume $\psi(t) \geqslant 0$ and there exist constants C_3 and C_4 such that*

$$\psi(n\tau) \leqslant C_3 + C_4 \tau \sum_{k=0}^{n-1} \psi(k\tau), \qquad 0 \leqslant n\tau \leqslant T;$$

then we have

$$\psi(n\tau) \leqslant C_3 \cdot e^{C_4 T}, \qquad 0 \leqslant n\tau \leqslant T.$$

Proof. Consider ordinary differential equation

$$\varphi'(t) = C_4 \varphi(t), \qquad \varphi(0) = C_3.$$

Its exact solution is $\varphi(t) = C_3 \cdot C^{c_4 t}$. It follows from $\varphi(t) \geqslant 0$,

$$\begin{aligned}\varphi(t+\tau) - \varphi(t) &= \tau\varphi'(t) + \frac{\tau^2}{2} \varphi''(t+\theta\tau)\\ &= \tau\varphi'(t) + \frac{\tau^2}{2} c^2 \varphi(t+\theta\tau) \geqslant \tau \cdot \varphi'(t).\end{aligned}$$

Thus,

$$\varphi(n\tau)-\varphi(0)=\sum_{k=0}^{n-1}[\varphi((k+1)\tau)-\varphi(k\tau)]$$
$$\geqslant \tau\sum_{k=0}^{n-1}\varphi'(k\tau)\geqslant \tau\cdot C_4\cdot\sum_{k=0}^{n-1}\varphi(k\tau)$$
$$\varphi(n\tau)\geqslant C_3+\tau\cdot C_4\cdot\sum_{k=0}^{n-1}\varphi(k\tau).$$

Now, we are to prove by contradiction that $\varphi(n\tau)\geqslant\psi(n\tau)$. Assume

$$\psi(n\tau)\begin{cases}\leqslant\varphi(n\tau), & n<n_1\leqslant[T/\tau],\\ >\varphi(n\tau), & n=n_1.\end{cases}$$

Then

$$\varphi(n_1\tau)-\psi(n_1\tau)\geqslant C_3+\tau C_4\sum_{k=0}^{n_1-1}\varphi(k\tau)-C_3-\tau C_4\sum_{k=0}^{n_1-1}\psi(k\tau)\geqslant 0.$$

This is in contradiction to $\psi(n_1\tau)>\varphi(n_1\tau)$. Therefore, the lemma is proved.

LEMMA 3. *Assume* $U_0(x)\in H_0^1[0,x_A]$; *then there is the estimate for the solution of the difference scheme* (2.5)–(2.7),

$$\|U^n\|\leqslant C_0,\qquad \|U_x^n\|\leqslant C_0,\qquad \|U^n\|_\infty\leqslant C_0,$$

where C_0 *is a constant independent of* h *and* τ.

Proof. Without loss of generality, we can assume that h is chosen so small that there are

$$\|U^0\|^2=h\sum_{j=1}^{J-1}[U_0(x_j)]^2$$
$$\leqslant 2\int_0^{x_L}|U_0(x)|^2\,dx=2\,\|U_0\|_{L_2}^2,$$
$$\|U_x^0\|^2=h\sum_{j=1}^{J-1}\left|\frac{U_0(x_{j+1})-U_0(x_j)}{h}\right|^2$$
$$\leqslant 2\int_0^{x_L}\left|\frac{dU_0(x)}{dx}\right|^2dx=2\left\|\frac{dU_0(x)}{dx}\right\|_{L_2}^2.$$

Thus, the conservation formula (2.9) yields

$$\|U^n\|\leqslant\text{Const.},\qquad \|U_x^n\|\leqslant\text{Const.}$$

It follows from the Lemma 1 that

$$\|U^n\|_\infty \leqslant \text{Const.}$$

THEOREM 1. *Assume* $U_0(x)\in H_0^1[0, x_L]$ *and for the solution of the problem* (2.1)–(2.3), $U(x, t)\in C^{(4,3)}$. *Then the solution of the conservative scheme* (2.5)–(2.7) *converges to the solution of the problem* (2.1)–(2.3) *with order* $O(\tau^2+h^2)$ *by* L_∞ *norm.*

Proof. Substituting the solution of the differential equation $U(x, t)$ into the scheme (2.5) and making Taylor's expansion, we have from (2.5)

$$\begin{aligned}&[U(jh, (n+1)\tau)]_t+\{\tfrac{1}{2}[u(jh, (n+1)\tau)+U(jh, n\tau)]\}_{\hat{x}}\\&\quad-\gamma^{-2}[U(jh, (n+1)\tau)]_{x\bar{x}t}+\frac{\beta}{3}\Big\{\frac{1}{2}[U(jh, (n+1)\tau)+U(jh, n\tau)]\\&\quad\cdot\frac{1}{2}\cdot[U(jh, (n+1)\tau)+U(jh, n\tau)]_{\hat{x}}+\frac{1}{4}[(U(jh, (n+1)\tau)+U(jh, n\tau))^2]_{\hat{x}}\Big\}\\&=R_j^n, \qquad 1\leqslant j\leqslant J-1, n=0, 1, 2, ...,\end{aligned} \tag{4.1}$$

where R_j^n is the truncation error and $\max_{1\leqslant j\leqslant J-1}|R_j^n|\leqslant \text{Const}\cdot(\tau^2+h^2)$, $0\leqslant n\tau\leqslant T$. Let $\varepsilon_j^n=U(jh, n\tau)-U_j^n$; then it follows from (2.5) and (4.1) that

$$\begin{aligned}&(\varepsilon_j^{n+1})_t+(\varepsilon_j^{n+1/2})_{\hat{x}}-\gamma^{-2}(\varepsilon_j^{n+1})_{x\bar{x}t}\\&\quad+\frac{\beta}{3}\Big\{\frac{1}{4}[U(jh, (n+1)\tau)+U(jh, n\tau)]\cdot[U(jh, (n+1)\tau)+U(jh, n\tau)]_{\hat{x}}\\&\quad+\frac{1}{4}[(U(jh, (n+1)\tau)+U(jh, n\tau))^2]_{\hat{x}}\\&\quad-U_j^{n+1/2}\cdot(U_j^{n+1/2})_{\hat{x}}-[(U_j^{n+1/2})^2]_{\hat{x}}\Big\}=R_j^n,\end{aligned} \tag{4.2}$$

$$\varepsilon_0^n=0, \qquad \varepsilon_J^n=0, \tag{4.3}$$

$$\varepsilon_j^0=0. \tag{4.4}$$

In view of difference properties and boundary condition, we have

$$\sum_{j=1}^{J-1}(\varepsilon_j^{n+1/2})_{\hat{x}}\cdot\varepsilon_j^{n+1/2}=0,$$

$$\sum_{j=1}^{J-1}(\varepsilon_j^{n+1})_{x\bar{x}t}\cdot\varepsilon_j^{n+1/2}=-\frac{1}{2\tau}\{[(\varepsilon_j^{n+1})_x]^2-[(\varepsilon_j^n)_x]^2\},$$

$$\sum_{j=1}^{J-1} \left\{\frac{1}{4}\left[U(jh,(n+1)\tau)+U(jh,n\tau)\right]\cdot\left[U(jh,(n+1)\tau)+U(jh,n\tau)\right]_{\hat{x}} - U_j^{n+1/2}\cdot(U_j^{n+1/2})_{\hat{x}}\right\}\cdot\varepsilon_j^{n+1/2}$$

$$= \sum_{j=1}^{J-1} (\varepsilon_j^{n+1/2})^2\cdot(U_j^{n+1/2})_{\hat{x}} + \sum_{j=1}^{J-1}\frac{1}{2}\left[U(jh,(n+1)\tau)+U(jh,n\tau)\right]\cdot(\varepsilon_j^{n+1/2})_{\hat{x}}\cdot\varepsilon_j^{n+1/2}$$

$$= -\sum_{j=1}^{J-1}\left[(\varepsilon_j^{n+1/2})^2\right]_{\hat{x}}\cdot U_j^{n+1/2} - \sum_{j=1}^{J-1}\left\{\frac{1}{2}\left[U(jh,(n+1)\tau)+U(jh,n\tau)\right]\cdot\varepsilon_j^{n+1/2}\right\}_{\hat{x}}\cdot\varepsilon_j^{n+1/2},$$

$$\sum_{j=1}^{J-1}\left\{\frac{1}{4}\left[(U(jh,(n+1)\tau)+U(jh,n\tau))^2\right]_{\hat{x}} - \left[(U_j^{n+1/2})^2\right]_{\hat{x}}\right\}\cdot\varepsilon_j^{n+1/2}$$

$$= \sum_{j=1}^{J-1}\left\{\frac{1}{2}\left[U(jh,(n+1)\tau)+U(jh,n\tau)\right]\cdot\varepsilon_j^{n+1/2}\right\}_{\hat{x}}\cdot\varepsilon_j^{n+1/2} + \sum_{j=1}^{J-1}\left[U_j^{n+1/2}\cdot\varepsilon_j^{n+1/2}\right]_{\hat{x}}\cdot\varepsilon_j^{n+1/2}$$

$$= \sum_{j=1}^{J-1}\left\{\frac{1}{2}\left[U(jh,(n+1)\tau)+U(jh,n\tau)\right]\cdot\varepsilon_j^{n+1/2}\right\}_{\hat{x}}\cdot\varepsilon_j^{n+1/2} - \sum_{j=1}^{J-1}(\varepsilon_j^{n+1/2})_{\hat{x}}\cdot\varepsilon_j^{n+1/2}\cdot U_j^{n+1/2}.$$

Multiplying (4.2) by $2\varepsilon_j^{n+1/2}$, summing it up for j and using the above deduction, we obtain

$$\frac{1}{\tau}\sum_{j=1}^{J-1}\left[(\varepsilon_j^{n+1})^2-(\varepsilon_j^n)^2\right] + \gamma^{-2}\cdot\frac{1}{\tau}\sum_{j=1}^{J-1}\left\{\left[(\varepsilon_j^{n+1})_x\right]^2-\left[(\varepsilon_j^n)_x\right]^2\right\}$$
$$+\frac{2\cdot\beta}{3}\left\{-\sum_{j=1}^{J-1}\left[(\varepsilon_j^{n+1/2})^2\right]_{\hat{x}}\cdot U_j^{n+1/2} - \sum_{j=1}^{J-1}(\varepsilon_j^{n+1/2})_{\hat{x}}\cdot\varepsilon_j^{n+1/2}\cdot U_j^{n+1/2}\right\}$$
$$= 2\cdot\sum_{j=1}^{J-1} R_j^n\cdot\varepsilon_j^{n+1/2};$$

i.e.,

$$\|\varepsilon^{n+1}\|^2+\gamma^{-2}\cdot\|\varepsilon_x^{n+1}\|^2$$
$$=\|\varepsilon^n\|^2+\gamma^{-2}\|\varepsilon_x^n\|^2+2h\tau\sum_{j=1}^{J-1}R_j^n\cdot\varepsilon_j^{n+1/2}$$
$$+\frac{2\beta}{3}h\tau\left[\sum_{j=1}^{J-1}(\varepsilon_j^{n+1/2})_{\hat{x}}(\varepsilon_{j+1}^{n+1/2}+\varepsilon_{j-1}^{n+1/2})\cdot U_j^{n+1/2}+\sum_{j=1}^{J-1}(\varepsilon_j^{n+1/2})_{\hat{x}}\cdot\varepsilon_j^{n+1/2}\cdot U_j^{n+1/2}\right]. \tag{4.5}$$

It follows from Lemma 3 and the Schwarz inequality that

$$\begin{aligned}\|\varepsilon^{n+1}\|^2+\gamma^{-2}\|\varepsilon_x^{n+1}\|^2\leqslant{}&\|\varepsilon^n\|^2+\gamma^{-2}\|\varepsilon_x^n\|^2\\&+\tau[\|R^n\|^2+\tfrac{1}{2}\|\varepsilon^{n+1}\|^2+\tfrac{1}{2}\|\varepsilon^n\|^2]\\&+\tfrac{1}{2}\beta\tau C_0[\|\varepsilon_x^{n+1}\|^2+\|\varepsilon_x^n\|^2+\|\varepsilon^{n+1}\|^2+\|\varepsilon^n\|^2.\end{aligned}$$

Let $E_\varepsilon^{n+1}\equiv\|\varepsilon^{n+1}\|^2+\|\varepsilon_x^{n+1}\|^2$; then

$$E_\varepsilon^{n+1}\leqslant E_\varepsilon^0+T\cdot\max_{0\leqslant k\leqslant[T\cdot\tau]}\|R^k\|^2+C_{01}\tau\sum_{k=1}^{n+1}E_\varepsilon^k. \tag{4.6}$$

Assume $C_{01}\cdot\tau\leqslant\frac{1}{2}$; then there is

$$E_\varepsilon^{n+1}\leqslant 2E_\varepsilon^0+2T\cdot\max_{0\leqslant k\leqslant[T/\tau]}\|R^k\|^2+2C_{01}\tau\sum_{k=1}^{n}E_\varepsilon^k.$$

It follows from (4.4) and Lemma 2 that

$$E_\varepsilon^n\leqslant 2T\cdot\max_{0\leqslant k\leqslant[T/\tau]}\|R^k\|^2\cdot C^{2\cdot C_{01}\cdot T}\leqslant\text{Const}\cdot(\tau^2+h^2)^2.$$

We have, in view of Lemma 1,

$$\|E_\varepsilon^n\|_\infty\leqslant\text{Const}\cdot(\tau^2+h^2).$$

It yields convergence.

THEOREM 2. *Assume $U_0(x)\in H_0^1[0,x_L]$ and τ is chosen properly; then the conservative scheme (2.5)–(2.7) is stable for the initial value by L_∞ norm.*

Proof. Suppose there are solutions fo the difference equations U_j^n and $\bar{U}_j^n$, which satisfy both the difference scheme (2.5) and the boundary condition (2.6). But, $U_j^0=U_0(x_j)$, $\bar{U}_j^0=\bar{U}_0(x_j)$. Let $\varepsilon_j^n=U_j^n-\bar{U}_j^n$. Similarly to the proof of Theorem 1, we can establish equations and the initial condition satisfied by ε_j^n and obtain

$$E_\varepsilon^{n+1}\leqslant E_\varepsilon^0+C_{01}\cdot\tau\sum_{k=0}^{n+1}E_\varepsilon^k.$$

Assume $C_{01} \cdot \tau \leqslant \frac{1}{2}$; then there is

$$\|E_\varepsilon^n\|_\infty \leqslant \text{Const} \cdot \|E_\varepsilon^0\|_\infty;$$

i.e., the difference scheme is stable.

(ii) *Comparison with Crank–Nicolson Scheme*

The first scheme one might consider using to solve problem (2.1)–(2.3) is the Crank–Nicolson (C-N) scheme:

$$(U_j^{n+1})_{\bar{t}} - \gamma^{-2} \cdot (U_j^{n+1})_{x\bar{x}\bar{t}} + \tfrac{1}{2}(1+\beta U_j^{n+1})(U_j^{n+1})_{\hat{x}} + \tfrac{1}{2}(1+\beta U_j^n)(U_j^n)_{\hat{x}} = 0. \quad (4.7)$$

We compare the conservation scheme (2.5) with the C-N scheme (4.7), through computing the initial value problem (2.10), (2.11). Taking $\beta = 1$, $\gamma = 1$, and $U_0(x) = 3 \cdot \text{Sech}^2((\sqrt{2}/4)(x-20))$, the exact solution of a single solitary wave described by

$$U(x \cdot t) = 3\ \text{Sech}^2\left(\frac{\sqrt{2}}{4}(x-20) - \frac{\sqrt{2}}{2}t\right). \quad (4.8)$$

TABLE VI

Computational Results at $t = 100$ and $h = 0.5$

τ	Method \ Results	CPU time (s)	N	A Value	A Error (%)	V Value	V Error (%)	cq_1 Value	cq_1 Error (%)	cq_2 Value	cq_2 Error (%)
	Exact solution (4.8)			3.0000		2.0000		16.9705		37.0996	
0.5	Conservative scheme (2.5)	134.6	8	2.9088	−3.04	1.9600	−2.00	16.9708	0.0017	37.1173	0.0478
	C-N scheme (4.7)	139.6	10	2.8736	−4.21	1.9550	−2.25	16.9709	0.0024	37.2129	0.3055
1	Conservative Scheme (2.5)	99.7	12	2.7792	−7.36	1.9100	−4.50	16.9713	0.0047	37.1585	0.1589
	C-N scheme (4.7)	133.1	19	2.6812	−10.63	1.9050	−4.75	16.9732	0.0159	38.5434	3.8917
2	Conservative scheme (2.5)	71.3	17	2.4597	−18.01	1.8100	−9.50	16.9698	−0.0041	37.2049	0.2838
	C-N scheme (4.7)		∞								

Note. "Error" stands for relative error. N denotes the number of iterations in one step time. A and V are amplitude and velocity, respectively. The iterative algorithm of the C-N scheme (4.7) is divergent for $\tau = 2$. The "cq_1" and "cq_2" are two conservative quantities:

$$cq_1 = h \sum_{j=1}^{J-1} U_j^n,$$

$$cq_2 = h \sum_{j=0}^{J-1} \left[\left(\frac{U_{j+1}^n + U_j^n}{2}\right)^2 + \frac{(U_{j+1}^n - U_j^n)^2}{h}\right].$$

The computational results for the solution (4.8) are given in Table VI. In computation, the C-N scheme (4.7) is solved by an iterative method which is similar to the method (2.14)–(2.17) for the conservative scheme (2.5).

Table VI demonstrates that the conservative scheme (2.5) can keep two conservative quantities and is more accurate than the C-N scheme (4.7). Furthermore, scheme (2.5) requires fewer iterations and less CPU time than the C-N scheme. Hence, the conservative scheme (2.5) is more efficient than the C-N scheme (4.7).

(iii) *Convergence of Iterations*

In order to solve the scheme (2.5), the iterative method (2.14)–(2.17) was given in the Section 2. Convergence of the iterative method dependent on the step sizes h and τ. But, our practical computation shows that the iterative method is convergent for various step sizes, which can be taken to ensure necessary accuracy of the approximation solution. The number of iterations is 3–8 for $h \leqslant 0.5$ and $\tau \leqslant 0.5$ and 17 for $h = 0.5$ and $\tau = 2$. Therefore, the iterative method (2.14)–(2.17) is suitable and efficient.

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